NST Part IA Physics
Easter 2024

Gravitational and Electromagnetic Fields

Examples Sheet, Synopsis and Recommended Books

1. Show that the gravitational field strength vector $\underline{\mathbf{g}}$ inside a uniform sphere of radius a and density ρ is given by:

$$\underline{\mathbf{g}} = -\frac{4\pi}{3} G\rho r \underline{\hat{\mathbf{r}}}$$

where r is the distance from the centre of the sphere and $\underline{\hat{\mathbf{r}}}$ is the unit radial vector.

Modelling the Earth as a solid, uniform sphere, show that a particle dropped into a smooth tunnel drilled through the Earth along a diameter will undergo simple harmonic motion and calculate the period of oscillation. (The mean density of the Earth is about 5500 kg m^{-3}). Compare the period of oscillation with the orbital period of a satellite in low Earth orbit.

2. A homogeneous sphere of radius a contains a spherical cavity of radius $a/4$ whose centre is 3a/8 from the surface. The diameter passing through the centres of the sphere and cavity meets the surface at points A and B. Show that the gravitational field at A and B are in the ratio 169:189.

3. A pendulum clock, known to keep time extremely accurately, is placed in a large room the floor of which has been covered with a uniform layer of lead of density ρ and thickness d.

   (a) Use dimensional analysis to show that the increase in the gravitational field strength at the clock due to the layer of lead is proportional to $Gd\rho$.

   (b) Show that the increase in the gravitational field strength, Δg, is given by $\Delta g = 2\pi G d\rho$.

   (c) After one year, the clock is found to be in error by one second. Is it fast or slow? Since the fractional changes in both the gravitational field strength and the period are very small, use an approximation to deduce the thickness of the layer of lead. (The density of lead is about 11,300 kg m^{-3}).

4. A system consists of two masses, m_1 and m_2 which interact under gravity. Explain what is meant by the statement that the gravitational potential energy of the system is given by

$$\Phi = -\frac{Gm_1m_2}{r}$$

including a discussion of the meaning of the minus sign, where r is the separation of the masses.

Two bodies each of mass M are fixed at points (a, 0, 0) and (-a, 0, 0), and a small mass m is placed at the origin.

(a) At first, the small mass is free to move only along the x-axis. Show that, when it is displaced to the position (x,0,0), its gravitational potential energy $\Phi(x)$ is given by

$$\Phi(x) = -GmM\left(\frac{1}{|a+x|} + \frac{1}{|a-x|}\right)$$

Sketch this potential function, and then find the first and second derivatives of Φ with respect to x to demonstrate that the mass is in unstable equilibrium at the origin.

(b) The arrangement is now changed so that the small mass is free to move only along the y-axis. Find its gravitational potential energy $\Phi(y)$ at position (0, y, 0) and sketch this potential function. Explain why the position (0, 0, 0) is now one of stable equilibrium.

5. An artificial satellite is in circular orbit around the moon at radius $R = a\ r$ where r is the radius of the moon itself. A short burn of the satellite's motor provides an impulse which halves the satellite's speed without changing its direction, and this alters the orbit to one which just grazes the Moon's surface. Use Kepler's first law to infer the shape and orientation of the orbit after the burn of the satellite's motor, and deduce the value of a .

6. The star S2 orbits the apparent black hole, Sgr A*, at the centre of our galaxy. S2's recently measured orbital parameters are:

   semi-major axis a = 140 light hours, period T = 15.9 years, eccentricity e = 0.883 .

   Use Kepler's third law to estimate the mass of the black hole Sgr A*, give your answer as a multiple of the solar mass. Also estimate the maximum velocity of S2 in its orbit. [Hint: angular momentum is a key parameter of orbits – try working via L]

7. An initially diffuse cold cloud of hydrogen gas condenses under gravitational attraction to form a compact spherically symmetric star, of mass and radius similar to our present day sun. Assuming for simplicity that the resulting star has constant density, estimate the gravitational potential energy released. If all the released energy was used to heat the condensing cloud and no mass was lost from the cloud, estimate the average temperature

of the final star. [You may assume each atom has thermal energy $\frac{3}{2}k_BT$ where k_B is Boltzmann's constant. Star formation is a much more complex process than this simple thought experiment might imply.]

8. Use the table of planetary data (including Pluto and Eris) given in lectures to plot a graph to check Kepler's third law. Use your graph to estimate the mass of the sun. Does the orbit of Eris (with high eccentricity) appear to be consistent with the major planets?

Electrostatics

9. Write down expressions for the electric field and electric potential at distance r from a point charge q. Calculate (expressing all energies in electron volts):

(a) the electric potential established by the nucleus of a hydrogen atom at the average distance of the circulating electron in its ground state ($r = 5.29\ 10^{-11}$ m);

(b) the electric potential energy of the atom when the electron is at this radius;

(c) the kinetic energy and total energy of the electron, assuming it to be in a classical circular orbit at this radius;

(d) the energy required to ionise the hydrogen atom.

10. Four carbon nuclei are at the apices of a regular tetrahedron of side length 0.154 nm. What is the net electrostatic force exerted on one nucleus by the other three?

11. (a) A charge q is uniformly distributed along the circumference of a thin ring of radius r. By summing the fields from each element of charge, show that the electric field at a point on the axis at a distance z from the plane of the ring is given by

$$\underline{\mathbf{E}} = \frac{qz}{4\pi\varepsilon_0(z^2 + r^2)^{3/2}}\underline{\hat{\mathbf{z}}}$$

(b) The charged ring is replaced by a circular sheet of charge having radius a and surface charge density σ. The circular sheet can be divided into infinitesimal rings of radius r and thickness dr. Write down an expression for the charge on each ring and hence, using the result in part (a), obtain an expression for the electric field due to each ring on the axis at a distance z from the sheet. Thus, by integration, show that the electric field on the axis at a distance z from the sheet is given by

$$\underline{\mathbf{E}} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right)\underline{\hat{\mathbf{z}}}$$

(c) Derive the results above by considering instead the gradient of the potential at a point on the axis.

(d) For the circular sheet of charge, show that, for $z > 0$:

$$\underline{\mathbf{E}} = \frac{\sigma}{2\varepsilon_0}\underline{\hat{\mathbf{z}}} \quad \text{for } z \ll a$$

$$\underline{\mathbf{E}} = \frac{Q}{4\pi\varepsilon_0 z^2}\hat{\underline{\mathbf{z}}} \quad \text{for } z \gg a$$

where $Q = \pi a^2 \sigma$ is the total charge. Interpret these results.

Conductors

12. A capacitor consists of two thin infinite concentric cylinders with inner and outer radii a and b.

   (a) Show that the capacitance, C, per unit length is $C = \frac{2\pi\varepsilon_0}{\ln(b/a)}$

   (b) At what radius does the electric field attain its maximum value, E_{max}? Show that the potential difference, V, between the inner and outer conductors can be expressed in terms of E_{max} as $V = aE_{max}\ln(b/a)$

   (c) If the outer cylinder has radius 10 mm and the breakdown electric field of air is $3\mathrm{V\mu m^{-1}}$, what radius should be chosen for the inner cylinder in order to maximise the potential difference V and what is that maximum?

13. A two circular metal plates of radius 10 cm form a parallel plate capacitor, C, and are separated by an adjustable air gap of width d. There is an unknown fixed capacitor C′ in parallel with C. When the capacitors hold a total change Q the following voltages were measured as d was varied:

d (mm)	30	25	20	15	10	7.5	5.0	3.0	2.5	1.5
V (volts)	28.8	28.05	27.15	26.1	24.6	23.1	20.4	15.75	13.5	9.6

   Plot a suitable graph to verify the expected relationship between the d and V. Hence find the total charge Q on the capacitors and the value of C′.

14. (a) Show that the capacitance of an isolated conducting sphere of radius R is given by $C = 4\pi\epsilon_0 R$.

   (b) The outer conductor (radius b, length l) of a long cylindrical capacitor is earthed. The inner conductor (radius a, length l) is hollow, insulated and uncharged. A sphere of radius R is charged to a potential V far from any other bodies and is inserted inside the inner conductor of the cylindrical capacitor without touching it. (The length l of the capacitor is very much greater than R so end-effects may be neglected.) Draw diagrams showing the distributions of the induced charges and the $\underline{\mathbf{E}}$ field lines in two perpendicular planes through the centre of the sphere one parallel and one perpendicular to the axis of the cylinders. Show that the electric field strength at a radius $a < r < b$ is $E = 2RV/rl$ and hence find the potential of the inner cylinder. Do your answers depend on whether or not the sphere is on the axis of the cylinders?

15. A solder pad in a portable mp3 player is at 5 V above ground. A silver whisker is growing from the pad and has a tip with radius 5 μm. Estimate the E field very near this tip.

Electric Dipole

16. Electric charges $+q$ and $-q$ are positioned at $(0, 0, \frac{1}{2}a)$ and $(0, 0, -\frac{1}{2}a)$. Estimate the electric field **E** at the points $(0, 0, d)$ and $(d, 0, 0)$ directly from Coulomb's law for the case $d \gg a$ using a binomial expansion. Verify that your result agrees with the expected values for an electric dipole of strength qa. [Optionally, if you have time, try to check the field at $(d \sin q, 0, d \cos q)$, this is requires a little more algebra.]

Currents and Magnetic Fields

17. A coaxial cable consists of a solid inner conductor of radius a and an outer cylindrical conductor of inner and outer radii b and c. Uniformly distributed currents of equal magnitude I flow in opposite directions in the two conductors. Derive expressions for the magnetic field **B**(r) for each of the regions $0 < r < a, a < r < b, b < r < c$ and $r > c$.

    Sketch and label a graph showing how the B field strength varies as a function of radius.

18. (a) Write down the expression for the B field at a distance r from a long straight wire carrying a current I. Find the magnitude of **B** if $I = +1$ A and $r = 1$ m. At what distance from the wire is B equal to that of the Earth's field (typically ~5×10^{-5} T)? Ignoring the Earth's field, draw diagrams showing the **B** field lines near (i) this wire, (ii) a wire carrying a current of -1 A.

    (b) Two long straight parallel wires are separated by 1m. Find the magnitude of the **B** field at a point midway between them if they carry currents of (i) +1, +1 A, (ii) −1, −1 A, (iii) +1, −1 A. Draw diagrams showing the B field lines near the wires in each of the three cases. What are the forces per unit length between the wires in each of the three cases?

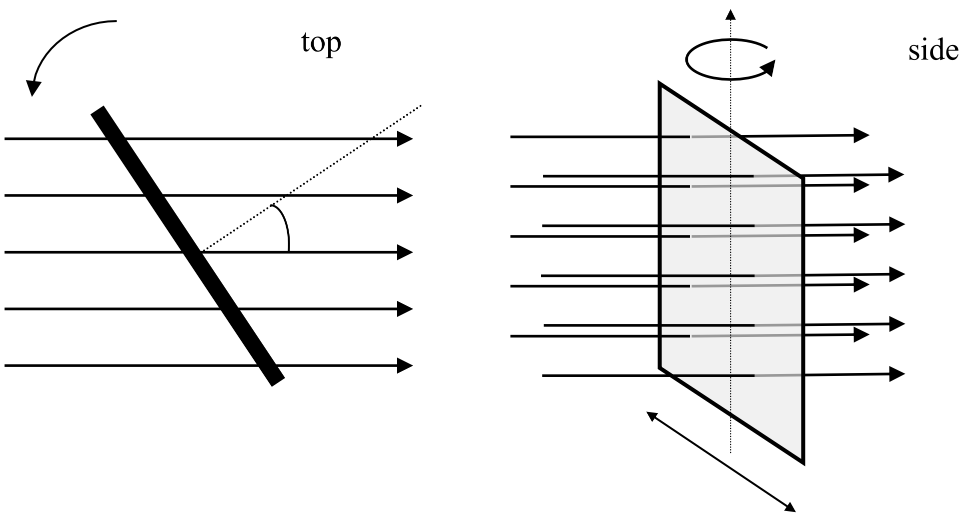
Electromagnetic Induction

19. Show that the inductance per unit length of two infinite parallel wires of radius a separated by a distance $2d$ ($2d \gg a$) is $L = \frac{\mu_0}{\pi} \ln {}^{2d}/_{a}$

20. (a) Derive an expression for the self-inductance of a solenoid of length l, with n turns per unit length and a circular cross section of radius r. (You should assume the B field is uniform along the length of the solenoid – *i.e.* ignore end effects.) If $l = 60$ cm, $n = 300$ turns per metre and $r = 2$ cm, calculate the value of the self inductance. What rate of change of current will produce an emf of 600 V across this solenoid?

    (b) Show that the energy stored in an inductor of self-inductance L carrying a current I is given by $\frac{1}{2}LI^2$. For the coaxial cable of question 17, find the magnetic energy density as a function of r. Write down expressions for the magnetic energy density stored in a cylindrical annulus of unit length, radius r and thickness $\mathrm{d}r$, in the regions $0 < r < a$ and $a < r < b$. Hence, by integration, find the magnetic energy stored per unit length of the cable and deduce the self-inductance per unit length. (You may assume that the thickness of the outer cylindrical conductor is very small so that you can ignore the magnetic energy stored in the region $b < r < c$.)

21. A square single turn coil with sides of length a is rotating at a constant angular speed ω about a vertical axis in a uniform horizontal $\underline{\mathbf{B}}$ field. The coil has resistance R.



(a) With the configuration shown in the diagram, is the magnetic flux through the loop increasing of decreasing? Hence use Lenz's law to deduce, for this configuration, the direction in which the current will be flowing in the coil. Find the directions of the forces on each of the four sides of the coil and show that there is a torque on the coil in the direction required by Lenz's law. Energy is dissipated as the current flows round the coil because of the resistance of the coil. Where does this energy come from?

(b) Write down an expression for the magnetic flux linked with the coil when the normal to the coil makes and angle $\theta\,(=\omega t)$ with the $\underline{\mathbf{B}}$ field; hence deduce that the current, I, flowing in the coil is given by

$$I = \frac{\omega a^2 B \sin \omega t}{R}$$

(c) Find the magnitudes of the forces on the arms QS and PT and hence show that the couple, G, required to keep the coil rotating at a constant angular speed is given by

$$G = \frac{\omega a^4 B^2 \sin^2 \omega t}{R}$$

(d) Find the instantaneous power produced by this couple and show that it is equal to the instantaneous power dissipated as heat in the coil.

Motion in Electric and Magnetic Fields

22. A mass spectrometer is used to measure the masses of different species of ions. Ions of charge q are produced essentially at rest and accelerated by a potential difference V. They then pass through a slit into a region in which there is a uniform magnetic field $\underline{\mathbf{B}}$ orientated perpendicular to the ions' direction of motion. In the field, the ions move in a semicircle, striking a photographic plate at a distance x from the entry slit. Show that the ion mass m is given by

$$m = \frac{B^2 q}{8V} x^2$$

Calculate the distance Δx between the spots on the photographic plate for a beam of singly ionised chlorine atoms of masses 35.0 and 37.0 m_u, if $V = 7.33$kV and $B = 520$ mT. [The unified atomic mass constant m_u=1.66 x 10^{-27}kg.]

23. When a homogeneous beam of electrons is passed through an evacuated region where there are simultaneously present an electric field $\underline{\mathbf{E}}$ of 30 V mm^{-1} and a magnetic field $\underline{\mathbf{B}}$ of 3.0 mT, it is found that the electrons are not deflected. When the $\underline{\mathbf{B}}$ field alone is present, the electrons move in a circle of radius 19 mm. Draw a diagram showing the relative orientations of the velocity and field vectors. Calculate the speed of the electrons and the ratio of their charge to mass.

A beam of protons is passed through the same apparatus. What would happen to this beam if the protons have (i) the same velocity as the electrons or (ii) the same kinetic energy as the electrons?

Electromagnetic Waves

24. Maxwell's equations in free space are:

$$\nabla \cdot \underline{\mathbf{E}} = 0, \quad \nabla \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}, \quad \nabla \cdot \underline{\mathbf{B}} = 0, \quad \nabla \times \underline{\mathbf{B}} = \mu_0 \varepsilon_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

Verify that a valid solution is given by:

$$\underline{\mathbf{E}}(x, y, z, t) = (E_x, 0, 0) \cos(kz - \omega t) \quad \text{and}$$
$$\underline{\mathbf{B}}(x, y, z, t) = \left(0, B_y, 0\right) \cos(kz - \omega t)$$

provided $\frac{E_x}{B_y} = \frac{\omega}{k}$, and $\frac{\omega^2}{k^2} = \frac{1}{\mu_0 \varepsilon_0}$. Discuss the physical meaning of this solution as fully as possible. What is the corresponding solution if $(kz - \omega t)$ is replaced by $(kz + \omega t)$?

Numerical Answers to Problems

1. ~85 mins.
3. 0.131 m.
5, $a = 7$.
6. mass ~$4\times 10^6 M_e$, max speed ~ 7800 km s^{-1} !
7. T ~ 9×10^6 K.
9. (a) 27.2 V, (b) −27.2 eV, (c) 13.6 eV, −13.6 eV, (d) 13.6 eV.
10. 8.5×10^{-7} N
12. (b) $r = a$, (c) 3.7 mm and 11 kV.
13. Q = 2.55×10^{-9} C, and C′ = 76 pF
14. (b) $2V(R / l)\ln(b / a)$.
15. ~1×10^6 V m^{-1}.
18. (a) 2×10^{-7} T, 4 mm; (b) 0, 0, 8×10^{-7} T, 2×10^{-7} N m^{-1}, like attractive, unlike repulsive.
20. (a) 85 μH, 7.0×10^6 A s^{-1}.
22. 7.9 mm.
23. 1.0 x 10^7 s^{-1}.

Gravitational and Electromagnetic Fields - Synopsis

F. Malte Grosche 2024

Gravitation: Newton's law, measurement of G. Action at a distance and concept of a local force field. Properties of conservative fields, including potential energy as path integral. Superposition of fields. Gauss law for gravity with simple quantitative applications.

Orbits: Kepler's laws. Derivation of elliptical orbits for planetary motion from Newton's law. Simple orbital calculations. Qualitative examples of gravity at work including tidal forces.

Electrostatic Fields: Static electricity, Coulomb's Law for point charges, the electric field **E** and corresponding potential for point charges and electric dipoles. Gauss' law for electrostatic fields. Properties of ideal conductors. Capacitance including calculation for simple geometries. Energy in a capacitor and energy in electric field. Mention effects of dielectric materials on capacitance and dipole moment of water molecule.

Magnetic Fields: Properties of bar magnets. Magnetic flux density **B**. Magnetic dipoles and currents as sources of **B**. Ampère and Biot-Savart laws, calculation of **B** field in simple cases. Lorentz force and motion of charged particles in electric and magnetic fields; J.J. Thomson's experiment. Faraday's law of induction; self and mutual inductance, energy stored in **B** field.

Maxwell's Equations: Displacement current term, Integral and differential statements. Example of plane wave solutions.

BOOKS

Understanding Physics (Second Edition), Mansfield M & O'Sullivan C (Wiley 2011)
Physics for Scientists and Engineers (Extended Version), Tipler P A & Mosca G (6th Edition,
Freeman 2008)
Fundamentals of Physics (Extended Edition), Halliday, D., Resnick, R. & Walker, J. (8th Edition, Wiley 2008).
The Elements of Physics, Grant I S and Phillips W R (OUP 2001).
Feynman Lectures on Physics Feynman R P et al. (Addison–Wesley 1963)

Note older editions of these books, might be available cheaply and would also be suitable for this course.